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On Markov modelling of turbulence

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We consider Lagrangian stochastic modelling of the relative motion of two fluid particles in the inertial range of a turbulent flow. Eulerian analysis of such modelling corresponds to an equation for the Eulerian probability distribution of velocity-vector increments which introduces a hierarchy of constraints for making the model consistent with results from the theory of locally isotropic turbulence. A nonlinear Markov process is presented, which is able to satisfy exactly, in the statistical sense, incompressibility, the exact results on the third-order structure function, and the experimental second-order statistics. The corresponding equation for the Eulerian probability density of velocity-vector increments is solved numerically. Numerical results show non-Gaussian statistics of the one-dimensional Lagrangian probability distributions, and a complex shape of the three-dimensional Eulerian probability density function. The latter is then compared with existing experimental data.

1. Introduction

The motion of fluid particles in a turbulent flow has been investigated by many authors by using the theory of continuous stochastic processes; this has proven to be a simple and powerful modelling technique. It has been extensively applied to describe the motion of one single particle in an inhomogeneous and unsteady large-scale turbulent field, the solution giving results on the mean concentration of a passive scalar advected by the turbulent flow. Analyses of statistical turbulence, a stochastic description, and recent developments related to such a field can be found in Monin & Yaglom (1971, 1975), Durbin (1983), van Dop, Nieuwstad & Hunt (1985), Sawford (1986), Novikov (1986), Thomson (1987), and references therein. Such models consider the velocity of a fluid particle as a stochastic process subjected to a large-scale sweeping and a random acceleration. They are based on the fact that the acceleration of a fluid particle is local in time and space, and can be assumed to be an uncorrelated random forcing. This also implies that the very small-scales viscous interaction is simply neglected, assuming the Reynolds number is sufficiently large.

When we extend the theory to consider the motion of a pair of particles, the modelling can be seen as the superposition of a relative motion and the motion of one single particle, or the particle centroid (Gifford 1959; Novikov 1966; Durbin 1980; Thomson 1990). The single particle, or particle centroid, contribution is modelled as described above for a single-particle system. The relative motion reflects more directly the internal turbulent structure, because of the appearance of an internal lengthscale (particle distance), and its description permits the introduction of concepts developed within the theory of turbulence (Monin & Yaglom 1975). The description of the motion of particle pairs can give information on the concentration fluctuation, as well as its mean value. Recent results on particle-pair modelling are reported in Durbin (1980), Sawford & Hunt (1986), Thomson (1990).

All these models are purely statistical, in the sense that the equations of motion are not used; moreover, Lagrangian measurements are still very difficult to perform and the accuracy of the models can be inferred only from few global results.

In the present work, we focus attention on the modelling of the relative motion of two fluid particles in the inertial range of an incompressible turbulent flow. Such modelling can then be coupled with a single particle description when a complete particle-pair model is required. The choice of relative motion in the inertial range is because it represents the Lagrangian counterpart of the Eulerian theory of locally homogeneous and isotropic turbulence (Monin & Yaglom 1975), expressed in terms of velocity-vector increments, whose results will be applied directly.

The relative motion of a pair of particle is modelled as a stochastic Markov process. The Markovian assumption cannot, at present, be derived from the equation of motion. Nevertheless, this assumption is consistent with experimentally observed similarity laws in Lagrangian and Eulerian descriptions (Monin & Yaglom 1975), and, as will be shown below, is able to give the correct value, derived by the equation of motion, for the third-order Eulerian structure function, and the exponential asymptotic behaviour of probability distributions supported by experimental and numerical data.

It is shown that a Lagrangian Markov model for relative velocity can be associated with an equation for the Eulerian probability distribution for velocity-vector increments between two points. From this equation we can obtain a hierarchy of constraints which correspond to satisfying the Eulerian results on three-dimensional statistical moments of any order. Particular attention will be given to incompressibility condition, and to Kolmogorov's (1941*b*) exact result on the third-order structure function derived directly from the Navier–Stokes equations. The effect of intermittency on Eulerian statistics and on turbulent diffusion is expressed in terms of the statistics of the dissipation fields, and is considered thorough the analysis.

In relation with the presented theory, a nonlinear Markov model is introduced and numerical results are presented. Such modelling is an extension of that introduced by Novikov (1989). A similar model was introduced previously by Thomson (1986; reported also in Thomson 1990), and, with more substantial differences, by Durbin (1980); these models were derived by similarity scaling arguments. On the other hand, one aim of this paper is to ensure an Eulerian quantitative consistency with Lagrangian stochastic models.

Regarding another aspect of the problem, it is of great interest to reproduce the probability distribution of velocity differences between two fixed spatial points, for the inertial range of turbulent flows, because it represents in many respects an exhaustive representation of turbulence small-scale statistics.

In the present work, the partial differential equation for the Eulerian probability density function of three-dimensional velocity increments corresponding to stochastic modelling is derived. Such an equation, when written in its general form, is composed of a transport term and a particle acceleration operator. An appropriate form of the operator can be constructed, independently from Markov modelling, for matching with real physics. We solve the equation corresponding to the Markov model proposed, analytically for the asymptotic cases and numerically for the complete solution.

In §2, the Lagrangian approach to the stochastic modelling is introduced, and in §3 these arguments are transferred to an Eulerian description, and, using the structure-functions formalism, the modelling constraints are derived. In §4, a Markov model is presented, and in §5 its Eulerian counterpart is analysed. Sections 6 and 7 present and discuss, respectively, the Eulerian and Lagrangian numerical results. In the Appendix we report, for completeness, the proof of the theorem which permits the connection between Lagrangian and Eulerian descriptions of turbulence.

2. Markovian modelling and Lagrangian statistics

Let us consider the relative motion of two fluid particles, in the inertial range of a turbulent flow, as a stochastic process for the Lagrangian trajectories. The stochastic differential equation for the relative motion (Novikov 1989; Thomson 1986) can be written as

$$\mathrm{d}\boldsymbol{u} = \boldsymbol{a}(t, \boldsymbol{r}, \boldsymbol{u})\,\mathrm{d}t + \mathrm{d}\boldsymbol{\mu}, \quad \mathrm{d}\boldsymbol{r} = \boldsymbol{u}\,\mathrm{d}t, \tag{1 a, b}$$

$\boldsymbol{u}(t)$ and $\boldsymbol{r}(t)$ being, respectively, the relative velocity and distance of the two fluid particles, and t is the time variable. The function $\boldsymbol{a}$ is the external acceleration, or relaxation term, due to the turbulent motion of scales larger than $r = |\boldsymbol{r}|$, the term $\mathrm{d}\boldsymbol{\mu}$ is the random forcing representing the jumps in the velocity due to the unresolved part of the turbulent flow, i.e. the underlying random turbulent motion which is not represented in the relaxation term. The statistics of $\mathrm{d}\boldsymbol{\mu}$ are given by the probability density $q(\boldsymbol{v}\,|\,\boldsymbol{u}, \boldsymbol{r})$ of a jump $\boldsymbol{v}$ in velocity, from the state $(\boldsymbol{u}, \boldsymbol{r})$, in a unit of time. Denoting

$$m^{(n)}_{i_1 i_2 \ldots i_n}(\boldsymbol{u}, \boldsymbol{r}) = \int v_{i_1} v_{i_2} \ldots v_{i_n}\, q(\boldsymbol{v}\,|\,\boldsymbol{u}, \boldsymbol{r})\,\mathrm{d}\boldsymbol{v}, \tag{2a}$$

we have

$$\overline{\mathrm{d}\mu_{i_1}\mathrm{d}\mu_{i_2}\ldots\mathrm{d}\mu_{i_n}} = m^{(n)}_{i_1 i_2 \ldots i_n}\,\mathrm{d}t + O(\mathrm{d}t^2), \tag{2b}$$

where the overbar stands for the probabilistic expected value. The mean value of the random forcing can be inserted into the relaxation term $\boldsymbol{a}$, and we assume, without loss of generality, that $m^{(1)}_i = 0$.

Assuming the system (1) to be Markovian it is possible (Novikov 1989; van Kampen 1981) to write the Chapman–Kolmogorov equation, and, from it, the master equation for the Lagrangian probability density $P_L(\boldsymbol{u}, \boldsymbol{r}\,|\,t, \boldsymbol{r}_0)$, the probability density at time t of the variables $\boldsymbol{u}(t)$ and $\boldsymbol{r}(t)$ given that $\boldsymbol{r}(0) = \boldsymbol{r}_0$. The master equation is written as

$$\frac{\partial P_L}{\partial t} + u_i \frac{\partial P_L}{\partial r_i} + \frac{\partial}{\partial u_i}(a_i P_L) = \int P_L(\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{r}\,|\,t, \boldsymbol{r}_0)\, q(\boldsymbol{v}\,|\,\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{r})\,\mathrm{d}\boldsymbol{v}, \tag{3}$$

where the left-hand side is the continuity equation in phase space, $(\partial P_L/\partial t) + \boldsymbol{\nabla}\cdot(\boldsymbol{F}P_L)$, the operator $\boldsymbol{\nabla}$ acting on the 6D space $\{\boldsymbol{r}, \boldsymbol{u}\}$ and the forcing $\boldsymbol{F}$ given by $[\boldsymbol{u}, \boldsymbol{a}]$. The initial condition for equation (3) is given by $P_L(\boldsymbol{u}, \boldsymbol{r}\,|\,0, \boldsymbol{r}_0) = P_E(\boldsymbol{u}\,|\,t, \boldsymbol{r})\,\delta(\boldsymbol{r}-\boldsymbol{r}_0)$, where $P_E(\boldsymbol{u}\,|\,t, \boldsymbol{r})$ is the Eulerian probability density at time t of the relative velocity $\boldsymbol{u}$ between two fixed points at distance $\boldsymbol{r}$.

Using smooth-$\boldsymbol{u}$ dependence of P_L and q, equation (3) can be expanded in Taylor series, obtaining the Kramers–Moyal expansion for the Lagrangian probability density function

$$\frac{\partial P_L}{\partial t} + u_i \frac{\partial P_L}{\partial r_i} + \frac{\partial}{\partial u_i}(a_i P_L) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial u_{i_1}\ldots\partial u_{i_n}}\left(m^{(n)}_{i_1 i_2 \ldots i_n} P_L\right). \tag{4a}$$

For Gaussian forcing the derivation is different than for jump processes (see details in van Kampen 1981), but we can formally put $m^{(n)}_{\ldots} = 0$ for $n > 2$, obtaining the correct equation for this case

$$\frac{\partial P_L}{\partial t} + u_i \frac{\partial P_L}{\partial r_i} + \frac{\partial}{\partial u_i}(a_i P_L) = \frac{1}{2}\frac{\partial^2}{\partial u_i \partial u_j}(m^{(2)}_{ij} P_L), \tag{4b}$$

which is known as forward Kolmogorov, or Fokker–Planck, equation.

From (4) we can derive the general equations for statistical characteristics of the flow. Multiplying (4) by $\mathscr{R}(\boldsymbol{r})\mathscr{U}(\boldsymbol{u})$, where $\mathscr{R}$ and $\mathscr{U}$ are arbitrary tensor functions, integrating over $(\boldsymbol{r},\boldsymbol{u})$ and using integration in parts, we obtain

$$\frac{\partial}{\partial t}\langle\mathscr{R}\mathscr{U}\rangle_L-\left\langle\frac{\partial\mathscr{R}}{\partial r_i}u_i\mathscr{U}\right\rangle_L-\left\langle\mathscr{R}\frac{\partial\mathscr{U}}{\partial u_i}a_i\right\rangle_L=\sum_{n=2}^{\infty}\frac{1}{n!}\left\langle\mathscr{R}\frac{\partial^n\mathscr{U}}{\partial u_{i_1}\ldots\partial u_{i_n}}m^{(n)}_{i_1i_2\ldots i_n}\right\rangle_L, \tag{5}$$

where the operation $\langle\ \rangle_L$ means averaging over the Lagrangian ensemble of trajectories. This equation is equivalent to (4), which can be obtained from (5) by choosing δ-functions for $\mathscr{R}$, $\mathscr{U}$ (see the Appendix and van Kampen 1981). At the same time, for various choices of $\mathscr{R}$, $\mathscr{U}$, we can get from (5) a hierarchy of equations for the evolutions of the moments of $\boldsymbol{u}(t)$ and $\boldsymbol{r}(t)$ and more general statistical characteristics.

For physical reasons, particular interest is generally devoted to the second-order statistics of $r^2=|\boldsymbol{r}|^2=r_ir_i$ and $u^2=|\boldsymbol{u}|^2=u_iu_i$. From (5), assuming $\mathscr{R}=r^2$ and $\mathscr{U}=1$; $\mathscr{R}=r_i$ and $\mathscr{U}=u_i$; $\mathscr{R}=1$ and $\mathscr{U}=u^2$, we get, respectively,

$$\frac{\partial}{\partial t}\langle r^2\rangle_L=2\langle u_ir_i\rangle_L, \tag{6a}$$

$$\frac{\partial}{\partial t}\langle u_ir_i\rangle_L=\langle u^2\rangle_L+\langle a_ir_i\rangle_L, \tag{6b}$$

$$\frac{\partial}{\partial t}\langle u^2\rangle_L=2\langle u_ia_i\rangle_L+\langle m^{(2)}_{ii}\rangle_L, \tag{6c}$$

where the fact that for locally homogeneous and isotopic turbulence $\langle u_i\rangle_L=0$ has been used. In this case, the solutions of (6) give (Monin & Yaglom 1975), asymptotically for $\epsilon t^3\gg r_0^2$,

$$\langle r^2\rangle_L=A_r\epsilon t^3,\quad \langle u_ir_i\rangle_L=A_{ur}\epsilon t^2,\quad \langle u^2\rangle_L=A_u\epsilon t, \tag{7a–c}$$

where $A_r A_u A_{ur}$ are dimensionless coefficients and ϵ is the mean rate of energy dissipation (see also results from the numerical calculations reported in §7). Equations (7) correspond to the experimentally supported Richardson law (Monin & Yaglom 1975) and to Kolmogorov's (1941*a*) similarity arguments.

Equations (7) show that turbulent diffusion is accelerated with respect to a Brownian, molecular, diffusion where $r^2\sim t$, but turbulence can be seen as producing such a diffusion in the velocity space. For this reason, the first attempt to get (7) from the model (1) has been made by considering the relative velocity $\boldsymbol{u}$ evolving as a Brownian motion; this means $\boldsymbol{a}\equiv 0$, and Gaussian jumps with $m^{(2)}_{ij}=g\epsilon\delta_{ij}$, g being a dimensionless constant (Monin & Yaglom 1975). In this case we get from (6), for large times, $A_u=3A_r=3g$ (see also subsequent discussion in §7 and tables 1 and 3).

Equations for higher-order moments of $\boldsymbol{u}(t)$ and $\boldsymbol{r}(t)$ can be obtained from (5); in particular it is easy to obtain the purely kinematical relations

$$\frac{\partial}{\partial t}\langle r^n\rangle_L=n\langle u_ir_ir^{n-2}\rangle_L. \tag{8a}$$

The equation for the moments of the relative velocity involves the higher-order moments of the random forcing. In general, assuming $\mathscr{R}=1$ and $\mathscr{U}=u^n$ in (5), we have

$$\frac{\partial}{\partial t}\langle u^n\rangle_L=n\langle a_iu_iu^{n-2}\rangle_L+\sum_{k=2}^{\infty}\frac{1}{k!}\left\langle\frac{\partial^k u^n}{\partial u_{i_1}\ldots\partial u_{i_k}}m^{(k)}_{i_1i_2\ldots i_k}\right\rangle_L, \tag{8b}$$

which becomes, for Gaussian forcing,

$$\frac{\partial}{\partial t}\langle u^n\rangle_L = n\langle a_i u_i u^{n-2}\rangle_L + \tfrac{1}{2}n(n-2)\langle u^{n-4}u_i u_j m_{ij}^{(2)}\rangle_L + \tfrac{1}{2}n\langle u^{n-2}m_{ii}^{(2)}\rangle_L. \qquad (8\,c)$$

A proper non-Gaussian forcing will be related to the turbulent intermittency of the Lagrangian velocity field (Novikov 1989, 1990). For a description of Lagrangian variables, including intermittency, from the dimensional argument, we have the general scaling (Novikov 1989)

$$r(t) \sim (\epsilon_l t^3)^{1/2}, \quad u(t) \sim (\epsilon_l t)^{1/2}, \qquad (9)$$

where ϵ_l is the rate of energy dissipation averaged over a sphere of radius l (Kolmogorov 1962; Monin & Yaglom 1975), such that the statistical averaging $\langle\epsilon_l\rangle = \epsilon$. From the scaling law (9), using the measure $l^2(t) = \langle r^2\rangle_L$, we find that the Lagrangian moments take the form

$$\langle r^n\rangle_L \sim \langle\epsilon_l^{n/2}\rangle\, t^{3n/2}; \quad \langle u^n\rangle_L \sim \langle\epsilon_l^{n/2}\rangle\, t^{n/2}; \qquad (10\,a,b)$$

thus, the effect of intermittency is expressed in terms of the statistics of the dissipation field. Relation (10) can be made more explicit by using the scale similarity of breakdown coefficients. Breakdown coefficients are defined by

$$q_{r,l} = \epsilon_r/\epsilon_l, \qquad (11)$$

and are also referred to as scale-invariant multipliers (Novikov 1969*a*, 1971, 1990; Monin & Yaglom 1975; Chhabra & Sreenivasan 1992); their statistics are expressed in terms of a scalar function $\mu(p)$ as

$$\langle q_{r,l}^p\rangle = (l/r)^{\mu(p)}. \qquad (12)$$

Assuming $\epsilon_L = \epsilon$, where L is an appropriate turbulence integral scale, from (12) we can write

$$\langle\epsilon_l^p\rangle = \hat{C}_p\,\epsilon^p(L/l)^{\mu(p)}. \qquad (13)$$

Here the $\hat{C}_p$ are coefficients which may depend on the large-scale structure of the turbulent flow, and the function $\mu(p)$ represents the intermittency corrections to the classical similarity scaling. Using (13) the statistics of the Lagrangian characteristics can be written (Novikov 1990) in the usual formalism as

$$\langle r^n u^m\rangle_L \sim \epsilon^{\frac{n+m}{2}}\, t^{\frac{3n+m}{2}} \left(\frac{T}{t}\right)^{\frac{3}{2}\mu\left(\frac{n+m}{2}\right)}. \qquad (14)$$

Here $T \sim L^{2/3}\epsilon^{-1/3}$, is the integral timescale, and (14) may contain a constant depending on the large-scale motion; also, we used the obvious fact that $\mu(0) = 0$ and the condition $\mu(1) = 0$ (Novikov 1969*a*). A universal form of the intermittency correction function $\mu(p)$ is not known, at present, but several phenomenological models (Kolmogorov 1962; Meneveau & Sreenivasan 1987; Novikov 1990; Chhabra & Sreenivasan 1992; Saito 1992) can give good approximation.

Equation (14) constitute the basic scaling relation for Lagrangian variables including intermittency corrections. We want to point out that the Lagrangian second-order moments have a linear dependence on the mean rate of energy dissipation ϵ, and are not subjected to an intermittency correction. In this sense, they can have a universal nature playing the analogous role of the third-order moments in the Eulerian description.

3. Structure functions and Eulerian probability

The Eulerian description of the local scales of turbulence in physical space is well suited to the use of the difference in velocity between two points fixed in space. Statistical moments of such a difference are the structure functions of turbulence, which can be expressed in simple terms including the condition of incompressibility and of local homogeneity and isotropy.

Let us write the velocity difference $\boldsymbol{u}(\boldsymbol{r})$ between two points, fixed in space, separated by a distance $\boldsymbol{r}$, as

$$u_i(r) = u_r n_i + \tilde{u}_i, \tag{15}$$

where $n_i = r_i r^{-1}$ is the unit vector in the $\boldsymbol{r}$-direction, $u_r = u_i n_i$ is the longitudinal component of velocity, and $\tilde{u}_i = u_i - u_r n_i$ is the transversal velocity, which is a vector contained in the plane normal to $\boldsymbol{r}$.

The second-order structure function $\langle u_i u_j \rangle$, where the operation $\langle\ \rangle$ means Eulerian ensemble averaging, can be expressed, because of local isotropy, in terms of only two scalar functions of r:

$$\langle u_i u_j \rangle = A(r)\,\delta_{ij} + B(r)\,n_i n_j. \tag{16}$$

From (15), (16) it follows that longitudinal and transversal components of velocity are uncorrelated:

$$\langle u_r \tilde{u}_i \rangle = 0. \tag{17}$$

However, they are not statistically independent (see (23)). The two components are physically different, even simply because of incompressibility. Loosely speaking, a transversal velocity difference signifies the presence of a vortex, whereas a longitudinal difference dominates in a saddle (deformation) region or when one point falls into a streaming region.

The condition of incompressibility can be transformed, in the statistical sense, into the structure-function condition (Monin & Yaglom 1975)

$$\frac{\partial}{\partial r_i}\langle u_i u_j \rangle = 0, \tag{18}$$

which corresponds to the well-known relation $A' + B' + (2/r)\,B = 0$, with the prime denoting a scalar derivative. It can be shown (Novikov 1992) that condition (18) is not only necessary, but also a sufficient condition for incompressibility in the statistical sense.

From (16) and (18) we can write the general expression for the second-order structure function which satisfies incompressibility:

$$\langle u_i u_j \rangle = \left(\langle u_r^2 \rangle + \frac{r}{2}\frac{\mathrm{d}}{\mathrm{d}r}\langle u_r^2 \rangle\right)\delta_{ij} - \left(\frac{r}{2}\frac{\mathrm{d}}{\mathrm{d}r}\langle u_r^2 \rangle\right) n_i n_j \tag{19}$$

depending on a single scalar function of a scalar argument, the variance of the longitudinal increment of the velocity, $\langle u_r^2 \rangle(r)$.

The third-order structure function can be written, because of local homogeneity and isotropy, and incompressibility, in the general form

$$\langle u_i u_j u_k \rangle = -2(C + rC')(n_i \delta_{jk} + n_j \delta_{ik} + n_k \delta_{ij}) + 6(rC' - C)\,n_i n_j n_k, \tag{20}$$

where the scalar function $C(r)$ alone characterizes all the moments. For the third-order moment we have the Kolmogorov (1941 *b*) result

$$\langle u_r^3 \rangle = -\tfrac{4}{5}\epsilon r, \tag{21}$$

derived directly from the Navier–Stokes equations in the inertial range of decaying turbulence. The same result was obtained for statistically stationary turbulence with large-scale random forcing (Novikov 1964). From (21) we get for the function C in equation (20) $C(r) = \epsilon r/15$. Equation (20) can then be rewritten in the final form (Novikov 1989)

$$\langle u_i u_j u_k \rangle = -\tfrac{4}{15}\epsilon(r_i \delta_{jk} + r_j \delta_{ik} + r_k \delta_{ij}). \tag{22}$$

Equation (22) gives the third-order correlation between longitudinal and transversal velocities

$$\langle u_r \tilde{u}_i \tilde{u}_j \rangle = -\tfrac{4}{15}\epsilon r(\delta_{ij} - n_i n_j), \tag{23}$$

showing their statistical dependence.

The general expression for the structure functions of any order can be written by using the intermittency formalism introduced in the previous section. For the second-order longitudinal velocity variance, introduced in (19), we have

$$\langle u_r^2 \rangle = C_0(\epsilon r)^{2/3}(L/r)^{\mu(2/3)}. \tag{24}$$

The intermittency correction for second-order moments is small (indicatively, Meneveau & Sreenivasan's 1987 model gives $\mu(\frac{2}{3}) = -0.0271$); the coefficient C_0 has been estimated from experimental results, generally neglecting the intermittency correction, as $C_0 = 1.6$–2.4. High-order statistics cannot, in general, leave aside intermittency influence, and are expressed by the following scaling, which is obtained from (13) and the Kolmogorov relation $u(r) \sim (\epsilon_r r)^{1/3}$ (see (9)):

$$\langle u^n \rangle \sim (\epsilon r)^{n/3}\left(\frac{L}{r}\right)^{\mu(n/3)}. \tag{25}$$

Equation (21) is the only exact result for structural functions of a velocity field derived from the Navier–Stokes equations. This is because third-order moments have a linear dependence on the dissipation rate ϵ and are not subject to correction of the similarity law due to intermittency phenomena (Monin & Yaglom 1975).

A first attempt to correlate the stochastic model (1) to the Navier–Stokes equations has been presented by Novikov (1989). In order to do this, the relation between Lagrangian and Eulerian probability distribution functions (Novikov 1969*b*) can be used. Consider the Eulerian probability distribution density $P_E(\boldsymbol{u} \mid t, \boldsymbol{r})$, the probability density at time t of the velocity difference $\boldsymbol{u}$ between two fixed points at distance $\boldsymbol{r}$. We have the theorem

$$P_E(\boldsymbol{u} \mid t, \boldsymbol{r}) = \int P_L(\boldsymbol{u}, \boldsymbol{r} \mid t, \boldsymbol{r}_0)\, \mathrm{d}\boldsymbol{r}_0, \tag{26}$$

which is valid, in homogeneous turbulence, if the boundaries do not depend on the velocity field (i.e. it is not valid for free surface flow, or for motion into a smooth waterbag). A complete proof of the theorem (26), due to Novikov (1969*b*), is reported in the Appendix.

Using relation (26) we can integrate (4*a*) or (4*b*) over the initial particle separations $\boldsymbol{r}_0$ and obtain a corresponding equation for P_E which can be written as

$$\frac{\partial P_E}{\partial t} + u_i \frac{\partial P_E}{\partial r_i} + \frac{\partial}{\partial u_i}(\bar{\alpha}_i P_E) = 0, \tag{27}$$

where $\bar{\boldsymbol{\alpha}}$ is the operator of *relative acceleration conditionally averaged with a given velocity increment* $\boldsymbol{u}$. The first term in (27) is zero for statistically stationary turbulence.

The operator $\bar{\alpha}$ is introduced in order to write a general equation for the Eulerian probability density, independently from its derivation from a Markovian modelling. A similar procedure of conditional averaging was used in the description of the turbulent vorticity field (Novikov 1993). In correspondence with (4) the operator $\bar{\alpha}$ acts as

$$\bar{\alpha}_i f = a_i f - \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^{n-1}}{\partial u_{j_2} \ldots \partial u_{j_n}} (m^{(n)}_{ij_2 \ldots j_n} f). \tag{28}$$

From (27), multiplying by u_j and integrating over $\boldsymbol{u}$, we get

$$\frac{\partial}{\partial r_i} \langle u_i u_j \rangle = \langle \bar{\alpha}_j \rangle = 0, \tag{29}$$

where the incompressibility condition (18) is used. Equation (29) gives the incompressibility constraint on the operator $\bar{\alpha}$. Unconditionally averaging of this operator means averaging over the velocity increments.

Multiplying (27) by $u_j u_k$ and integrating we get the second equation of the hierarchy:

$$\frac{\partial}{\partial r_i} \langle u_i u_j u_k \rangle - \langle u_k \bar{\alpha}_j \rangle - \langle u_j \bar{\alpha}_k \rangle = 0. \tag{30}$$

Substitution of (22) gives

$$\tfrac{4}{3}\epsilon \delta_{ij} + \langle u_i \bar{\alpha}_j \rangle + \langle u_j \bar{\alpha}_i \rangle = 0. \tag{31a}$$

Equation (31*a*) is a further constraint on the operator $\bar{\alpha}$, which ensures that the probability distribution, solution of (27), satisfies the third-order moments result (22), consequent from the Navier–Stokes equations. With the definition (28) of the operator $\bar{\alpha}$, (31) then becomes

$$\tfrac{4}{3}\epsilon \delta_{ij} + \langle u_i a_j \rangle + \langle u_j a_i \rangle = -\langle m^{(2)}_{ij} \rangle. \tag{31b}$$

The model of pure diffusion ($\boldsymbol{a} \equiv 0$) in velocity space, discussed in Monin & Yaglom (1975, §2), satisfies condition (29) but does not satisfy the dynamical condition (31*b*). In particular, summation in (31*b*) over $i = j$ with $\boldsymbol{a} = 0$ gives $g = A_r = -\frac{4}{3} < 0$, which contradicts the physical sense of turbulent diffusion (7).

Equations for higher-order moments can be obtained by using an analogue of formula (5) in the Eulerian description. The next equation of the hierarchy reads

$$\frac{\partial}{\partial r_l} \langle u_l u_k u_i u_j \rangle - \langle u_i u_j \bar{\alpha}_k \rangle - \langle u_i u_k \bar{\alpha}_j \rangle - \langle u_j u_k \bar{\alpha}_i \rangle = 0; \tag{32}$$

this and subsequent ones can be used to connect the operator $\bar{\alpha}$ with higher-order moments of the velocity field, which are more influenced by the intermittency effects. In this case, less exact quantitative information is known, due also to the increasing complexity of the tensorial form of high-order moments. Nevertheless, any experimental or theoretical result can be handled in the same way as presented for second- and third-order moments, giving further constraints on the modelling and improving its representation of the real physics.

4. A Markov model

In order to construct a model, in the framework of the stochastic system (1), we must define the relaxation function $\boldsymbol{a}$, and the statistics of the random forcing $\mathrm{d}\boldsymbol{\mu}$. We consider here a generalization of the model introduced by Novikov (1989).

The function $\boldsymbol{a}$ is given by the ratio of the actual velocity value to a typical timescale $\boldsymbol{a} \sim -\boldsymbol{u}/\tau$, in such a way that the velocity, in the absence of forcing, is able to decay to zero (relax) with a timescale given by τ. For a three-dimensional flow, the relaxation can be written as

$$a_i = -\frac{c}{\tau(r)}(u_i + \gamma u_r n_i), \tag{33}$$

where c and γ are two dimensionless coefficients of the model. The parameter $\gamma \neq 0$ gives a different relaxation for longitudinal and transversal components of velocity. These components have different physical natures, and relaxation is in general non-isotropic. The instantaneous (local) timescale, $\tau(r)$, for two particles separated by a distance $\boldsymbol{r}$ can be defined by dimensional arguments:

$$\tau(r) = \epsilon^{-1}\langle u^2\rangle. \tag{34}$$

As a first approach, we assume the forcing as Gaussian distributed. In this case, the second-order tensor of forcing (diffusion in velocity space) has the general form

$$m_{ij} = m_{ij}^{(2)} = \epsilon(a n_i n_j + b(\delta_{ij} - n_i n_j)). \tag{35}$$

Here a and b are the non-dimensional coefficients of longitudinal and transversal forcing respectively (the coefficient a must not be confused with the relaxation function $a_i(\boldsymbol{u}, r)$).

In the Eulerian representation this corresponds to the operator

$$\bar{\alpha}_i = a_i - m_{ij}\frac{\partial}{\partial u_j}. \tag{36}$$

The incompressibility condition (29) is satisfied trivially. The dynamical condition (31) implies the following relations between relaxation and forcing coefficients:

$$(2+s)\,c = (3+s)(b+\tfrac{4}{3}), \quad (2\gamma - s)\,c = (3+s)(a-b). \tag{37}$$

Here, in general, $s = \frac{2}{3} - \mu(\frac{2}{3})$ is the exponent of the power law $\langle u^2\rangle \sim r^s$. Note that incompressibility condition (18) can be rewritten as

$$\langle \tilde{u}^2\rangle = (s+2)\langle u_r^2\rangle, \tag{38}$$

where $\tilde{u} = (\tilde{u}_i \tilde{u}_i)^{1/2}$ is the module of the transversal component of velocity.

The random forcing, representing the unresolved part of the turbulent flow, is isotropic only if $a = b$, which means $\gamma = \frac{1}{2}s$. This shows that the isotropy of relaxation and the isotropy of forcing are mutually exclusive, a result which is due to (31), and so to (22). Intuitively, it seems more reasonable, from a physical point of view, that forcing would be isotropic but we can give no proof in support of such an assumption.

The actual value of $\langle u^2\rangle$ appearing in (34) can be calculated by (19), using (24) with the experimentally estimated coefficient C_0. The value $C_0 \approx 2$ will be used in the numerical calculations.

5. Eulerian probability distribution equation

The Markov model introduced above corresponds to an Eulerian probability distribution for the vector of velocity increments which is a solution of the partial differential equation (27). Such a distribution satisfies the condition of incompressibility (18) and the dynamical result for the third-order moment (22) derived from the

Navier–Stokes equations. In this framework, Markov modelling can be used for the derivation of an Eulerian probability distribution consistent with theoretical and experimental results on the local structure of turbulence.

In general, the solution of (27), with different assumptions on the function $\bar{\alpha}$, gives the probability distribution of velocity-vector increments. The knowledge of such a distribution is important for advancing the theoretical description of turbulence, and for practical purposes as well. For example, the subgrid-scale modelling in large-eddy simulations requires the definition of the 'stress' tensor $\boldsymbol{\tau}$, which ultimately can be expressed in terms of velocity-vector increments:

$$\tau_{ij}(x) = \overline{V_i V_j} - \bar{V}_i \bar{V}_j = \frac{1}{2}\int u_i(x''-x')\,u_j(x''-x')f(x,x')f(x,x'')\,\mathrm{d}x''\,\mathrm{d}x', \qquad (39)$$

where V_i is the absolute velocity vector, $\bar{V}_i = \int V_i(x')f(x,x')\,\mathrm{d}x'$ is the filtered velocity field, and f is the filter function, determined by a numerical scheme. According to the concept of conditional averaging of the Navier–Stokes equations (Novikov 1993), we have to average the product of velocity increments in (39) conditionally with a fixed filtered velocity field. Also, the solution of (27) gives the Eulerian counterpart of Lagrangian diffusion models, such as the one introduced in the previous section, which, in this way, can be experimentally verified without involving Lagrangian measurements.

For locally homogeneous and isotropic turbulence $P_E(u\,|\,r)$ is generally a function of three independent variables, u_r, $\tilde{u}$, and r. In these variables, and by introducing the model of the previous section, (27) can be rewritten as

$$u_r\frac{\partial P_E}{\partial r}+\left(\frac{\tilde{u}^2}{r}-\frac{c(\gamma+1)}{\tau(r)}u_r\right)\frac{\partial P_E}{\partial u_r}-\left(\frac{u_r\tilde{u}}{r}+\frac{c}{\tau(r)}\tilde{u}\right)\frac{\partial P_E}{\partial \tilde{u}}-\frac{c}{\tau(r)}(\gamma+3)\,P_E$$
$$=\frac{\epsilon}{2}\left(a\frac{\partial^2}{\partial u_r^2}+b\frac{\partial^2}{\partial \tilde{u}^2}+\frac{b}{\tilde{u}}\frac{\partial}{\partial \tilde{u}}\right)P_E, \qquad (40)$$

with $\tau(r)$ defined by (34). We make the problem dimensionless normalizing velocities with $(\epsilon r)^{1/3}$. For further simplification, we neglect the intermittency correction and assume that the classic similarity scaling (Kolmogorov 1941 *a*, *b*) is valid. In this case the r-dependence of the probability distribution is completely included in the normalization term. With these assumptions the new dimensionless variables are

$$x = u_r(\epsilon r)^{-1/3}, \quad y = \tilde{u}(\epsilon r)^{-1/3}, \quad \mathscr{P}(x,y) = (\epsilon r)\,P_E(u_r,\tilde{u}\,|\,r), \qquad (41)$$

and the moments of the velocity differences can be then computed from

$$\langle x^n y^m\rangle = 2\pi\int_0^{+\infty}\mathrm{d}y\,y\int_{-\infty}^{+\infty}\mathrm{d}x\,x^n y^m\mathscr{P}(x,y), \qquad (42)$$

representing a linear integration along x, and axisymmetric integration along y.

The equation for the Eulerian probability distribution can be written in final form as

$$\left(y^2-\frac{x^2}{3}-\frac{c}{C_2}(\gamma+1)\,x\right)\frac{\partial\mathscr{P}}{\partial x}-\left(\tfrac{4}{3}xy+\frac{c}{C_2}y\right)\frac{\partial\mathscr{P}}{\partial y}-\left(x+\frac{c}{C_2}(\gamma+3)\right)\mathscr{P}$$
$$-\frac{a}{2}\frac{\partial^2\mathscr{P}}{\partial x^2}-\frac{b}{2}\frac{\partial^2\mathscr{P}}{\partial y^2}-\frac{b}{2y}\frac{\partial\mathscr{P}}{\partial y}=0, \qquad (43)$$

with boundary conditions

$$\left.\begin{array}{l}\mathscr{P}\to 0 \quad \text{when} \quad x\to\pm\infty,\\ \mathscr{P}\to 0 \quad \text{when} \quad y\to+\infty,\\ \partial\mathscr{P}/\partial y=0 \quad \text{at} \quad y=0,\end{array}\right\} \tag{44}$$

and the normalization condition

$$2\pi\int_0^{+\infty}\mathrm{d}y\,y\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathscr{P}=1, \tag{45}$$

which makes the system non-homogeneous. The constants a and b are given by (37) (with $s=\frac{2}{3}$). The parameter C_2 is derived from the definition of relaxation time (34), and is equal to $C_2=\langle x^2+y^2\rangle$, that is

$$C_2=2\pi\int_0^{+\infty}\mathrm{d}y\,y\int_{-\infty}^{+\infty}\mathrm{d}x(x^2+y^2)\mathscr{P}. \tag{46}$$

It is important to note that the parameter C_2 is determined by the solution $\mathscr{P}$ of (43), so condition (46) makes the complete system nonlinear and integro-differential. Eventually, the solution will be determined such that C_2 matches with its experimental value (see (19) and (24)) $C_2=\frac{11}{3}C_0=\frac{22}{3}$.

From the system (43)–(46) it is easy to obtain the moment equations. Multiplying (43) by 1, x, and x^2, and integrating, in the sense of

$$2\pi\int_0^{+\infty}\mathrm{d}y\,y\int_{-\infty}^{+\infty}\mathrm{d}x\ldots,$$

we get, respectively

$$\langle x\rangle=0,\quad \langle x^2\rangle-\tfrac{3}{8}\langle y^2\rangle=0,\quad 3\langle x^3\rangle-2\langle xy^2\rangle+\tfrac{4}{3}=0. \tag{47a–c}$$

The last two formulae correspond, respectively, to the incompressibility condition (38) and to the dynamical relation (22).

The left-hand side of (43) is composed of three physically different contributions: the last three terms represents *diffusion* in velocity space, the terms containing the constant c are the local *relaxation*, the others represents *transport* terms derived from equation (1b). The diffusion term is of order $\sim u^{-2}$ (where u indicates either x or y) and is dominant close to the origin, where the probability distribution reaches its maximum values. Transport is $\sim u$, and dominates asymptotically the tails of the distribution. Relaxation, ~ 1, is intermediate. From these estimates of relative contributions it is possible to solve the equation in the asymptotic cases when one or two of such contributions can be neglected.

Close to the origin, where $x^2+y^2\ll 1$, the transport term can be neglected from (43). The solution of the *diffusion/relaxation* equation is Gaussian:

$$\mathscr{P}(x,y)=\frac{c^{3/2}(\gamma+1)^{1/2}}{C_2^{3/2}\pi^{3/2}a^{1/2}b}\exp\left[-\frac{c}{C_2}\left(\frac{(\gamma+1)x^2}{a}+\frac{y^2}{b}\right)\right]. \tag{48}$$

In this case, the constants a and b must be evaluated without the $+\frac{4}{3}$ term in (37), because it derives in (31) from the transport term, which is now neglected, imposing the correct value for the third-order moments which are now zero, and otherwise condition (46) does not hold. This solution is symmetric with respect to the variable x and, even though it satisfies incompressibility, it does not satisfies the dynamical constraint on the third-order moment of the probability distribution.

In the *far asymptotic* case, when $x^2+y^2\gg 1$, we can consider the *pure transport* regime. This represents pure inertial motion under the condition of incompressibility,

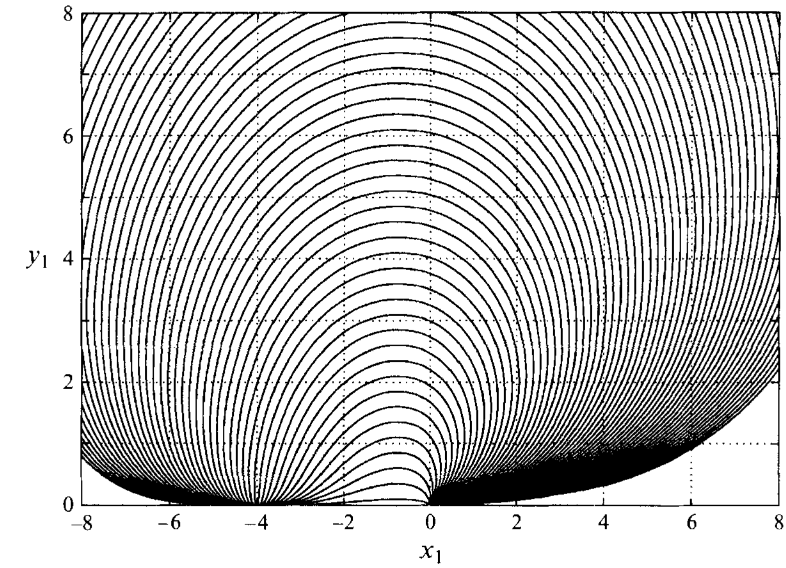

FIGURE 1. Characteristic curves of the Eulerian probability distribution equation in the asymptotic regime, when the diffusion contribution is neglected.

and does not depend on the choice of relaxation and forcing operators. This case has been studied in detail by Novikov (1992), where it is shown that the function

$$\mathscr{P}(x,y) = y^{-3/4}F\left(\frac{x^2+y^2}{y^{1/2}}\right) \tag{49}$$

is a solution for any function $F(\cdot)$. It is worth noting the unusual combination of the velocity components in the argument, which demonstrates the physical difference between longitudinal and transversal velocity increments.

In the *asymptotic* case, when still $x^2+y^2 \gg 1$ but only diffusion is neglected, we can consider the *no-diffusion* regime. This regime is also a solution of the complete equation (43) with $c = \frac{11}{6}$ and $\gamma = \frac{1}{3}$ (in this case (37) gives $a = b = 0$). Under this condition, (43) loses its elliptic character and becomes a first-order partial differential equation. Then the solution is obtained (up to an arbitrary function of a scalar argument) by integrating along the characteristic curves, starting from a curve in the (x,y)-plane where the solution is known, by using

$$\frac{\mathrm{d}x}{\left(y^2-\dfrac{x^2}{3}-c/C_2(\gamma+1)\,x\right)} = \frac{\mathrm{d}y}{-(\frac{4}{3}xy+c/C_2 y)} = \frac{\mathrm{d}\mathscr{P}}{(x+c/C_2(\gamma+3))\mathscr{P}}. \tag{50}$$

In this case, the variables can be scaled by c/C_2, eliminating c/C_2 from equation. Writing

$$x = \frac{c}{C_2}x_1, \quad y = \frac{c}{C_2}y_1, \quad \mathscr{P}(x,y) = \left(\frac{C_2}{c}\right)^3 \mathscr{P}_1(x_1,y_1),$$

and putting $\gamma = \frac{1}{3}$, we can write

$$\frac{\mathrm{d}x_1}{-y_1^2+\frac{1}{3}x_1^2+\frac{4}{3}x_1} = \frac{\mathrm{d}y_1}{\frac{4}{3}x_1y_1+y_1} = \frac{\mathrm{d}(\ln\mathscr{P}_1)}{-(x_1+\frac{10}{3})}; \tag{51}$$

the characteristic curves derived from the first equality in (51) are reported in figure 1. As we expected a focus is localized at the origin, owing to the elimination of the

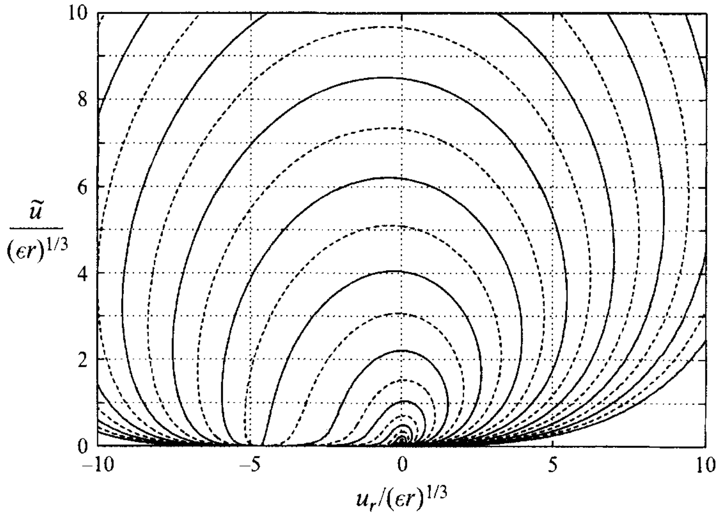

FIGURE 2. Three-dimensional Eulerian probability density function in the limit case in the absence of diffusion, and parameters $c = \frac{11}{6}, \gamma = \frac{1}{3}$. Horizontal and vertical axes represent, respectively, the longitudinal and the module of the transversal normalized velocity increments. Equi-probability curves are shown by continuous lines at levels $10^{-9}, 10^{-8}, \ldots, 10^{1}$, dashed lines are intermediate.

diffusive term. Also a marked asymmetry, with respect to longitudinal velocity, is observed as a consequence of the dynamical constrain (22), and another unexpected focus is found for a negative value of x_1. A solution of (50), for $c = \frac{11}{6}$ and $\gamma = \frac{1}{3}$, is determined by solving equation (51) with a fourth-order Runge–Kutta method along the characteristics, using stretched coordinates in order to have a proper resolution close to the origin, where an integrable singularity is present. The solution can be obtained by fixing the value of the function on one arbitrary curve which intersects all characteristics curves. The line $x_1 = -\frac{3}{4}$ has been chosen for numerical stability; the value of the solution on this line is fixed following the far-asymptotic solution (Novikov 1992). This gives $\mathscr{P}_1(-\frac{3}{4}, y_1) = K y_1^{-3/4} \exp(-y_1)$, determined from (49) to match the asymptotic shape of experimental measurements of a one-dimensional probability distribution of longitudinal velocity increments (Gagne, Hopfinger & Frisch 1988; Kailasnath, Sreenivasan & Stolovitzky 1992; Praskovsky 1992*a*). Then, when the solution in $\{x_1, y_1\}$ is obtained, we have to impose the condition (46), $C_2 = \langle x^2 + y^2 \rangle$. Combining this with the relation

$$\langle x^2 + y^2 \rangle = \left(\frac{c}{C_2}\right)^2 \langle x_1^2 + y_1^2 \rangle,$$

which is due to the scaling of (51), we obtain that the eventual value of C_2 is given by

$$C_2 = c^{2/3} \langle x_1^2 + y_1^2 \rangle^{1/3}. \tag{52}$$

By imposing the value $c = \frac{11}{3}$, and using (52), such a solution satisfies (43) and, consequently, represents a three-dimensional probability density function of velocity increments which satisfies incompressibility, Kolmogorov's (1941*b*) result on third-order moments, and classical similarity for all moments. In the present asymptotic case we get $C_2 = 1.65$, which is much less than its experimental value. The solution is reported in figure 2 in the (x, y)-plane, where equi-probability contours are plotted (continuous lines) at levels $10^{-9}, 10^{-8}, \ldots, 10^{1}$. We can observe that the vectorial shape

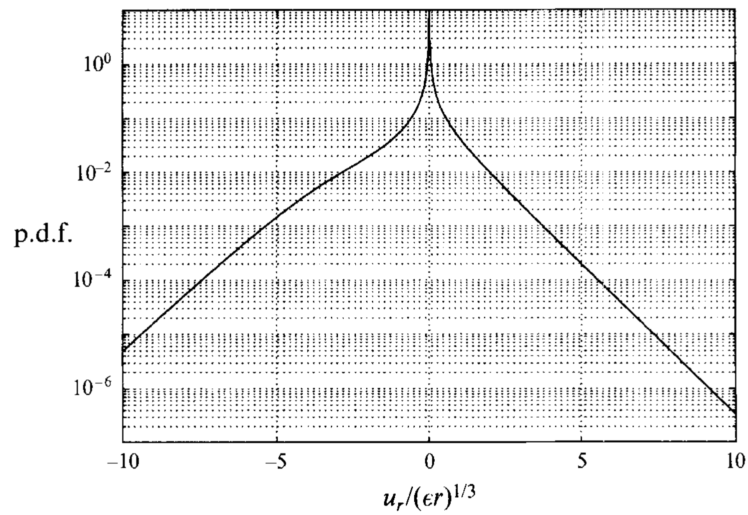

FIGURE 3. One-dimensional Eulerian probability density function of the longitudinal normalized velocity increment, in the limit case in the absence of diffusion, and parameters $c = \frac{11}{6}, \gamma = \frac{1}{3}$.

is very articulated, representing the non-trivial statistical dependence between longitudinal and transversal velocity increments, as introduced in (17) and (23). The one-dimensional probability density function of the longitudinal velocity is reported in figure 3; we can observe the unrealistic shape close to the origin, the asymmetry which gives the Kolmogorov law $\langle x^3 \rangle = -0.8$, and exponential tails. It is interesting to note that the ratio between the asymptotic slopes is around 1.4 as observed experimentally (Praskovsky 1992*a*). The Lagrangian counterpart of this limit situation loses physical sense. However, this Eulerian distribution has a meaning as an asymptotic result, which helps to perform the full solution of (43) to which next section is devoted.

A fundamental question that the Eulerian analysis can answer is about the *realizability* of the flow field corresponding to a given Lagrangian stochastic model. It means, in general, that we have to verify the existence of a solution of (27). By realizability, in this framework, we mean the existence of a solution $P_E(\boldsymbol{u} \,|\, \boldsymbol{r}) \geqslant 0$, which satisfies incompressibility (18) and the third-order tensor expression (22). In the present modelling, this corresponds to the existence of a solution $\mathscr{P}(x,y) \geqslant 0$ of the system (43)–(46). Also, from (52), for example in the characteristics solution, it can be seen that once the model parameters are fixed (in this case c) then the value of C_2 follows. Vice versa, if we fix a value for C_2, then only one value of the model parameter c realizes such condition. From this we can expect that, in the full solution of (43), once we fix C_2 to its experimental value, then the model parameters c and γ cannot take any value but will be determined in consequence. The Eulerian analysis permits the determination of the existence and the characteristics of the flow field assumed implicitly by stochastic models.

6. Eulerian numerical results

In this section, we consider the solution of the complete system (43)–(46) for the Eulerian probability density function. Firstly, we shall describe the numerical method adopted and then we shall present and comment the numerical results.

The system (43)–(46) is composed of different parts. Equation (43) is a homogeneous linear one with varying coefficients partial differential equation whose boundary conditions (44) are also homogeneous. The normalization condition (45) makes the system globally non-homogeneous, and the trivial solution not acceptable. Moreover, condition (46) makes the system integro-differential and nonlinear. This last condition creates a relation between the value of C_2 and the model parameters c and γ. The nonlinearity is managed by fixing *a priori* the value of C_2, and the value of γ. Then, the corresponding value of the parameter c, such that condition (46) is satisfied, is determined by an iterative procedure, in which, at each step, the linear system (43)–(45) is solved. The value of C_2 is fixed to its experimental value $C_2 = \frac{22}{3}$; then, for each γ, if the solution exists, we have a corresponding value of c.

The main problem is thus reduced to the solution of a linear partial differential equation (43). A steady solution such as that cannot be obtained as the asymptotic solution for large times of the corresponding unsteady equation (derivable from (27)). We verified numerically that this unsteady system does not tend to any asymptotic steady state. In fact, most of the arguments developed in §3 become meaningless when the flow is not statistically stationary, the transient evolution has no physical meaning and cannot drive to the actual steady solution; also this difficulty may be related to the unbounded domain in which (4*b*), even in the case of Eulerian stationarity, cannot reach a steady asymptotic solution. Equation (43) is a second-order elliptic equation. The coefficients of the second-order derivatives are constants whereas the coefficients of the remaining terms vary with x and y; in particular they grow (in absolute value) with the distance from the origin. It was shown in the asymptotic analyses of §5, that far from the origin the second-order terms become negligible so that the elliptic character of the equation is lost. When the $\{x, y\}$ space is made discrete, the linear partial differential equation (43) transforms onto a large algebraic linear system. The large dimension of the linear system makes the use of direct methods not applicable (if the x, y domain is transformed onto a 200×200 grid, then the linear system dimension is 40000×40000). The corresponding matrix is five-diagonal, non-symmetric, non-diagonal dominant, and badly conditioned reflecting the ill-behaviour described above. Actually, the matrix tends asymptotically, for large u, to be singular.

After an extensive series of numerical analyses with several iterative techniques which all failed to converge, it was found that the system could be solved by using the Bi-Conjugate Gradient Stabilized method (BiCGSTAB, van der Vorst 1991) combined with using incomplete decomposition (Meijerink & van der Vorst 1981) as preconditioning technique.

With this method the solution is found to be very stable and independent of the discretization parameters. The results presented here have been obtained in the finite domain $\{x, y\} \equiv \{(-30, 30) \times (0, 30)\}$ discretized on a 200×120 stretched grid. The same results were obtained on the same domain starting from a 60×40 grid, and in smaller domains, excluding boundary effects. In particular, as can be seen from the characteristics curves in figure 1, and from the solution in figure 2, boundary effects at large negative values of x are transported at smaller (in absolute value) values of positive x. The solution has been also verified, *a posteriori*, to satisfy (43) with a relative error smaller than 10^{-8} in the whole integration domain. The integral equations (47) have been verified (by two-dimensional numerical integration of the solution) with a maximum error smaller than 1%.

The parameter γ has been fixed, initially, to the value $\gamma = \frac{1}{3}$. This ensures the isotropy of the forcing term ($a = b$) which appears, physically, as the most reasonable first choice. We found that the appropriate parameter for ensuring condition (46) is given by $c = 28$. The three-dimensional probability density function $\mathscr{P}(x, y)$ is reported in figure 4. Equi-probability curves are plotted with continuous lines for $\mathscr{P} = 10^{-11}, 10^{-10}$,

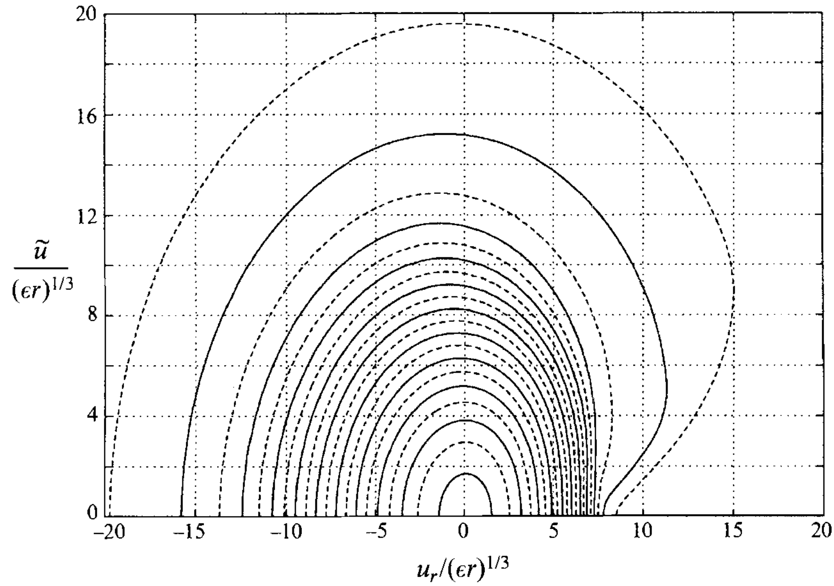

FIGURE 4. Three-dimensional Eulerian probability density function, with parameters $c = 28, \gamma = \frac{1}{3}$. Horizontal and vertical axes represent, respectively, the longitudinal and the module of the transversal normalized velocity increments. Equi-probability curves are shown by continuous lines at levels $10^{-11}, 10^{-10}, \ldots, 10^{-2}$, dashed lines are intermediate.

$\ldots, 10^{-2}$. At small values of u the distribution is close to Gaussian (contour lines are approximately ellipses) then, moving from the origin, the non-symmetric behaviour appears significant, and the tails of the distribution begin to take the asymptotic shape shown on figure 2. The vectorial shape contains the non-trivial dependence among different components of the velocity increments present in (19) and (22), including the statistical dependence between longitudinal and transversal components (17) and (23). Accurate experimental measurements of the three-dimensional probability distribution, which can be compared with the present results, are at present not available but are technically possible (Praskovsky 1992*b*; Thoroddsen & Van Atta 1992) and certainly needed for advancing the theory. The major difficulty in experimental comparison is given by the large amount of data required to build three- and even two-dimensional probability distributions with good resolution in the tails of the distribution. Work is currently in progress to realize extensive measurements. A large amount of data can also be obtained by high-resolution numerical simulation (Jimenez *et al.* 1993), even though these are still limited to small Reynolds numbers and do not present a substantial inertial range. In particular, in figure 5 we report the corresponding joint probability density of the longitudinal velocity increment and one transversal component $\tilde{u}_1$. This distribution is easier to measure, because it involves *only two* components of velocity increments in known directions.

In order to make some experimental comparisons, we report, in figure 6, the one-dimensional probability density function of the longitudinal component of velocity increments. Since from the theory we have the right moments $\langle x^2 \rangle = 2$ and $\langle x^3 \rangle = -0.8$, we can observe the asymmetry with respect to the corresponding Gaussian distribution (dashed lines), and asymptotic exponential tails. This distribution can be compared with detailed experimental measurements (Gagne *et al.* 1988; Kailasnath *et al.* 1992; Praskovsky 1992*a*) of the same function and with numerical results by direct simulation (Vincent & Meneguzzi 1991). The global shape differs from experiments in

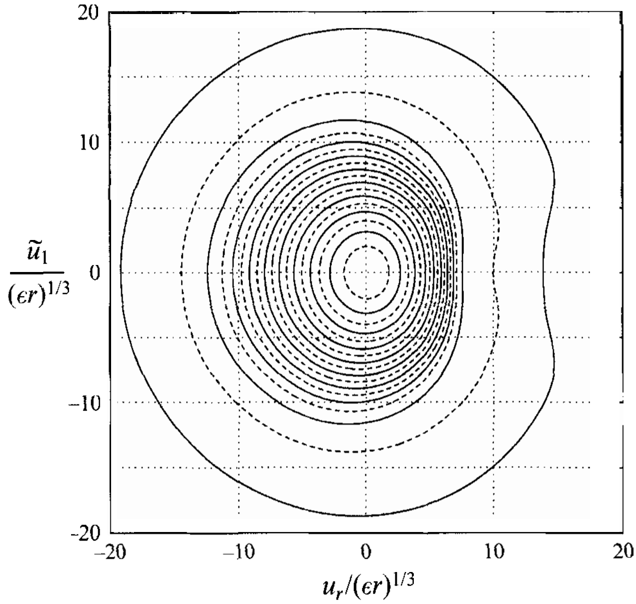

FIGURE 5. Two-dimensional Eulerian probability density function, with parameters $c = 28, \gamma = \frac{1}{3}$. Horizontal and vertical axes represent, respectively, the longitudinal and one component of the transversal normalized velocity increments. Equi-probability curves are shown by continuous lines at levels $10^{-10}, 10^{-9}, \ldots, 10^{2}$, dashed lines are intermediate.

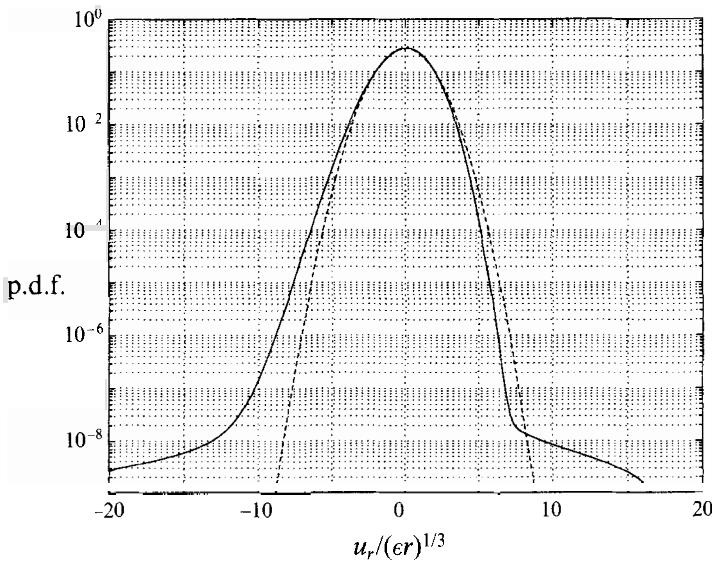

FIGURE 6. One-dimensional Eulerian probability density function of the longitudinal normalized velocity increment, with parameters $c = 28, \gamma = \frac{1}{3}$. The corresponding Gaussian distribution is shown by a dashed line.

the following aspects: this model gives a density which decays more rapidly at moderately large values of x, whereas the exponential tails, also present in experiments, decay slower (slopes being about one-third smaller than those of experimental tails). Also, experimental tails slopes are not unique, but depend on the distance r between

c	γ	$\langle x^4 \rangle$	$\langle x^5 \rangle$	$\langle x^6 \rangle$
28.0	$\frac{1}{3}$	12.79	−14.88	149.3
18.5	1	12.73	−14.69	144.9
12.5	2	12.64	−14.32	142.1
9.5	3	12.59	−14.12	140.6
6.5	5	12.53	−13.83	138.6

TABLE 1. Parameters used in the Eulerian numerical solutions, and one-dimensional statistics of the Eulerian probability density function for the longitudinal component of the velocity increment.

measuring points, reflecting the intermittency correction to similarity. The right and left slopes are in the same ratio (1.1–1.5) as in experiments. These descriptions are reflected quantitatively in the differences in the statistics. The fourth-order moment obtained is $\langle x^4 \rangle \approx 12.8$ while the measured experimental value is 20–30, similarly we get $\langle x^6 \rangle \approx 150$, compared to the experimental value 500–2000. On the other hand, because of smaller tail slopes, very high moments are expected to be larger than experiments.

The present model ensures theoretically the proper values for the statistics up to the third moments while higher-order statistics follow. In particular, the choice of Gaussian random forcing appears certainly to be reductive for the turbulence; nevertheless it represents the necessary first approach in the Markovian representation. The slopes of the exponential tails, and the values of high-order moments, depend on intermittency which we have not considered in this paper. Intermittency corrections can be included by using a more general forcing and, possibly, a modification of the relaxation term.

Several other pairs of the parameters c and γ have been tested to analyse the dependence of the model on them. In table 1 the different pairs of parameters, all corresponding to the same value of C_2, are reported, and the corresponding moments of order 4 to 6 of the longitudinal velocity difference are listed. From these figures we can see that such statistics do not depend significantly on these parameters. Actually, none of the solutions differs significantly, either in the three-dimensional shape or quantitatively (differences of a few percent) up to very low values of the probability, where smaller values of the parameter c correspond to slightly higher slopes in the tails. Moreover, we found that for γ smaller than about 0.3 the probability takes negative values, which means that for such parameters the Eulerian flow is not realizable. Also, we were not able to find the solution corresponding to the model of pure diffusion in velocity space discussed in Monin & Yaglom (1975, §2); this is not a proof but does cast doubt on the realizability of such a model.

7. Lagrangian numerical results

The results on the Lagrangian probability distributions have been obtained by simulation of trajectories from the Markov model described in §4. Time integration is performed with the Milshtein (1974) scheme which allows an $O(\Delta t)$ pathwise accuracy and $O(1/n)$ mean-square convergence, n being the number of trajectories reproduced (Pardoux & Talay 1985).

The problem is made dimensionless by normalizing with the initial separation of particles r_0, such that $\boldsymbol{r}(0) = r_0 \cdot [1\ 1\ 1]$, and the rate of dissipation ϵ. The initial relative velocity is not influential and fixed at $\boldsymbol{u}(0) = \boldsymbol{0}$. The integration time step has been

c	γ	A_r	A_u
28.0	$\frac{1}{3}$	0.260	3.45
18.5	1	0.207	2.89
9.5	3	0.122	1.97
6.5	5	0.083	1.50

TABLE 2. Parameters used in the Lagrangian simulations, and calculated global dispersion coefficients for the square module of particle distance, and of relative velocity.

chosen constant at $\Delta t = 0.05$ which has shown not to influence the final statistical results even when varied by one order of magnitude. The final statistics has been extracted at $t = 1000$ using $n = 10^5$ trajectories. The asymptotic behaviour given by (7), theoretically expected for $t \gg 1$, is observed for $t \geqslant 10$.

The global dispersion coefficients of the Richardson law in (7) are reported in table 2 for the Lagrangian simulations performed. As experimental estimate for the constant A_r has been given by Tatarski (1960) who obtained values ranging from 0.06 to 0.45, even though errors can be large. The value $A_r = 0.26$ obtained here is also comparable with the value $A_r = 0.1 \pm 0.05$ obtained in the simulations by Fung *et al.* (1992). These were obtained by using a simplified representation of a homogeneous turbulent flow as the superposition of unsteady random Fourier modes, such that incompressibility is automatically satisfied, and the energy over modes is distributed in agreement with second-order spatial statistics.

The Lagrangian one-dimensional probability density functions for one component of the particle separation r_x, of relative velocity u_x, and for the longitudinal velocity u_r, are reported in figure 7(*a*)–7(*c*), respectively, at $t = 1000$, in correspondence of the parameters $c = 28$, $\gamma = \frac{1}{3}$. These variables are normalized by dividing by the square roots of ϵt^3, ϵt, and ϵt, respectively, as dictated by (7). The corresponding Gaussian distributions are reported with dashed lines. In order to indicate quantitative information, the statistical characteristics up to sixth order are listed in table 3. The probability distributions are sensibly different from Gaussian. In particular the distribution of particle distance appears more similar to an exponential distribution, sharper than Gaussian at the origin, and with exponential tails. The probability distribution of relative velocity is closer to Gaussian, for small values, and decreases more slowly asymptotically. The tails appear close to exponential, but more detailed experiments may be necessary to correctly evaluate the behaviour. Qualitatively analogous behaviour has been observed for the other values of the parameters c and γ. It should be noticed that a preliminary simulation with $\gamma = 0$ (a value which appears forbidden from the Eulerian analysis) gave Gaussian distributions for r_x and u_x, whilst other trials with negative values of γ gave distributions with opposite behaviour (less sharp than Gaussian at the origin, and rapidly decaying tails).

It is of interest to notice the shape of the probability density for the longitudinal relative velocity, $u_r(t) = u_i(t)\, r_i(t)\, r^{-1}(t)$. It has a positive mean, and a positive skewness, contrary to the negative skewness of the Eulerian longitudinal velocity. This gives a possible picture of the turbulent dispersion process, in which fluid particles move approximately together for most of the time, and are impulsively separated when one particle falls into an internal jet-like motion.

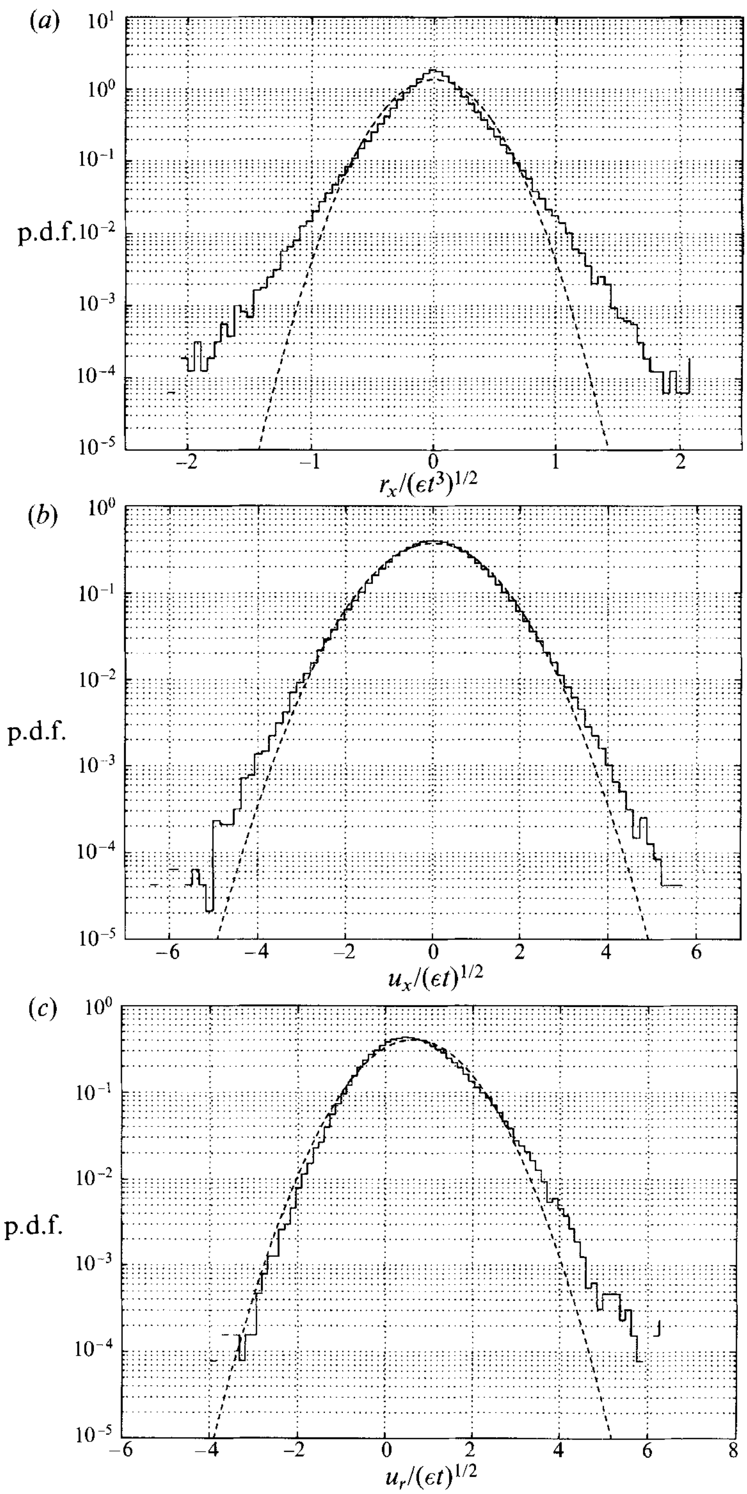

FIGURE 7. One-dimensional Lagrangian probability density function of one component of: (*a*) the vectorial distance between two fluid particles, (*b*) the relative velocity between two fluid particles, and (*c*) the longitudinal relative velocity between two fluid particles, with parameters $c = 28, \gamma = \frac{1}{3}$. The corresponding Gaussian distribution is shown by a dashed line.

n	$\langle (r_x/(\epsilon t^3))^n \rangle$	$\langle (u_x/(\epsilon t))^n \rangle$	$\langle (u_r/(\epsilon t))^n \rangle$
1	0	0.003	0.641
2	0.086	1.150	1.384
3	0	0.009	2.476
4	0.037	4.665	6.797
5	0.001	−0.018	18.597
6	0.037	36.380	63.020

TABLE 3. One-dimensional large-time asymptotic statistics for the Lagrangian probability distributions of one component of the particle distance, one component of the relative velocity, and longitudinal velocity, obtained from numerical simulation with parameters $c = 28, \gamma = \frac{1}{3}$.

8. Discussion

The representation of fluid particle motion by Lagrangian stochastic models is very attractive because of its simplicity and theoretical clarity, but it suffers from the difficulty of comparing the results with theoretical and experimental data. In the present work, an Eulerian representation has been derived in correspondence to the Lagrangian Markovian modelling for the relative motion of two particles in the inertial range of a turbulent flow. This approach permits the verification of the realizability of the corresponding Eulerian flow field, as well as the determination of its statistical characteristics which, in this way, can give information about the accuracy of the stochastic modelling without involving direct Lagrangian measurements. The choice of relative motion permits the introduction of results from the theory of locally homogeneous and isotropic turbulence. It is shown that stochastic models can be constructed in such a way that they will automatically satisfy known Eulerian results. In particular, a simple model, with local relaxation and Gaussian forcing, has been presented which contains the condition for incompressibility, classical similarity scaling, the Eulerian exact formula derived from the Navier–Stokes equation on the third-order structure function, and the Eulerian experimental data for the second-order moments. Numerical simulations of the Markovian model have shown that the Lagrangian probability distribution of particle separation and relative velocity departs significantly from Gaussian and presents exponential asymptotic tails. The longitudinal relative velocity distribution elongates towards positive values (positive skewness) while the opposite occurs with the Eulerian distribution. These results give a picture of an intermittent turbulent dispersion which may be related to the presence of internal, jet-like, turbulent structures.

The Eulerian analysis imposes a hierarchy of restrictions, which must be added to other global limitations (Thomson 1987), in the definition of a Lagrangian model which can be closer to representing the physics of real turbulence. Combination of such consistent modelling with single-particle, or particle-centroid, models, as extensively described by Durbin (1980) and Thomson (1990), will improve the representation of complete two-particle dispersion; this fact will permit the computation of the mean concentration and concentration variance of a passive scalar advected by turbulent motion. The model is focused on inertial-range turbulence, while extension to large-scale flow can be introduced using a large-scale relaxation time (Durbin 1980; Novikov 1986).

Also, in correspondence with the Lagrangian stochastic model, the partial differential equation of the three-dimensional Eulerian probability distribution of velocity increments has been obtained. The solution represents, statistically, the Eulerian flow in which the particles, moving as dictated by the stochastic process, are dispersed. This solution, found numerically, contains the condition of incompressibility and the complex statistical dependence among vectorial components of the velocity field up to third order.

The Eulerian probability distribution shows an articulated three-dimensional shape which we hope will stimulate experimental measurements. The one-dimensional probability distribution of the longitudinal velocity increments has been compared with existing data. The first three moments are in perfect agreement because they are satisfied exactly from the theory, higher-order moments are significantly smaller and the tails of the distribution, even though exponential, present an initially larger and asymptotically smaller slope than experiments, while the slopes of the right and left sides conserve the same ratio as experimentally observed. However, it has been shown that high-order statistics agreement, including intermittency corrections, can be introduced hierarchically in the model. In particular, the use of Gaussian forcing represents a starting point but is certainly inadequate to reproduce high-order statistics of turbulence. Improved modelling will have to consider a more general expression for random forcing, and, possibly, for the relaxation function, to be able to satisfy higher-order statistics which will take into account intermittency of the dissipation rate (Monin & Yaglom 1975; Novikov 1990) and other theoretical results from the Navier–Stokes equations (Novikov 1991, 1993). Also, data from extensive experimental measurements, and from high-resolution numerical simulations must be used in support and completion of the theoretical arguments.

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Appendix. Connection between Lagrangian and Eulerian description of turbulence (Novikov 1969*b*)

The probability distribution of a random variable can be expressed with the help of averaged δ-functions. The expression

$$P(x) = \langle \delta(x-y) \rangle \tag{A 1}$$

defines the probability density of the variable x, and the average operation is assumed over the ensemble of all possible realizations y. From (A 1) the moments of the variable y can be computed from

$$\langle y^n \rangle = \int x^n P(x)\,\mathrm{d}x = \left\langle \int x^n \delta(x-y)\,\mathrm{d}x \right\rangle. \tag{A 2}$$

Consider the Lagrangian joint probability distribution at time t of a variable $\boldsymbol{\sigma}$ and the positions $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$ of n fluid particles that are initially in positions $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(n)}$ respectively. It can be written as

$$P_L(\boldsymbol{\sigma}, \boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \dots, \boldsymbol{X}^{(n)} \mid t, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \dots, \boldsymbol{a}^{(n)}) = \left\langle \delta(\boldsymbol{S}_L(t, \boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(n)}) - \sigma) \prod_{m=1}^{n} \delta(\boldsymbol{y}^{(m)}(t, \boldsymbol{a}^{(m)}) - \boldsymbol{X}^{(m)}) \right\rangle, \tag{A 3}$$

where S_L represent the value assumed by the variable $\boldsymbol{\sigma}$ on fixed particles and $y^{(m)}$ corresponds to trajectories of these particles.

Under the hypotheses that (i) the fluid is incompressible, which means that the Jacobian

$$\frac{\mathfrak{D}y(t,a)}{\mathfrak{D}a} = 1, \tag{A 4}$$

and that (ii) the boundary of the flow is independent of the fluid motion we can prove that

$$\int P_L(\boldsymbol{\sigma}, \boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)} \,|\, t, \boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(n)}) \prod_{m=1}^{n} \mathrm{d}\boldsymbol{a}^{(m)} = P_E(\boldsymbol{\sigma} \,|\, t, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}), \tag{A 5}$$

where the right-hand side is the Eulerian probability distribution of the variable $\boldsymbol{\sigma}$ at time t dependent on the position $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}$ of n points fixed in space.

Theorem (A 5) states that the Eulerian probability density of a variable, at a certain time dependent on the position of points in space, is given by the integral of the Lagrangian probability over the initial positions of all fluid particles that at that time pass through such positions.

Proof. The Eulerian probability is defined by

$$P_E(\boldsymbol{\sigma} \,|\, t, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}) = \langle \delta(\boldsymbol{S}_E(t, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}) - \boldsymbol{\sigma}) \rangle, \tag{A 6}$$

where $\boldsymbol{S}_E$ represent the value assumed by the variable $\boldsymbol{\sigma}$ on fixed points. The relation between Lagrangian and Eulerian values of the variable is obvious:

$$\left. \begin{array}{c} S_L(t, \boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(n)}) = S_E(t, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}). \\ | \\ y^{(m)}(t, \boldsymbol{a}^{(m)}) = \boldsymbol{x}^{(m)}, \quad m = 1, \dots, n \end{array} \right\} \tag{A 7}$$

Rewrite the integral on the left-hand side of (A 5) using definition (A 3). The averaging operator can be put outside of the integral, because of the condition (ii). Changing the integration from $\mathrm{d}\boldsymbol{a}$ to $\mathrm{d}\boldsymbol{y}$ and using (A 3), (A 4) and (A 7), gives

$$\int P_L \prod_{m=1}^{n} \mathrm{d}\boldsymbol{a}^{(m)} = \left\langle \int \delta(\boldsymbol{S}_L - \boldsymbol{\sigma}) \prod_{m=1}^{n} \delta(\boldsymbol{y}^{(m)} - \boldsymbol{X}^{(m)}) \,\mathrm{d}\boldsymbol{y}^{(m)} \right\rangle = P_E. \qquad \square$$

A generalization for the case of variable density has been presented by Novikov (1986).

The same argument is valid when we consider the relative motion of two fluid particles.

Consider the relative position of two particles $r_i(t) = X_i(t, \boldsymbol{a}) - X_i(t, \boldsymbol{a}')$ and their relative velocity $u_i(t) = V_i(X_i(t, \boldsymbol{a})) - V_i(X_i(t, \boldsymbol{a}'))$, with $\boldsymbol{r}_0 = \boldsymbol{a} - \boldsymbol{a}'$. The theorem (A 5) can be rewritten as

$$\int P_L(\boldsymbol{u}, \boldsymbol{r} \,|\, t, \boldsymbol{r}_0) \,\mathrm{d}\boldsymbol{r}_0 = P_E(\boldsymbol{u} \,|\, t, \boldsymbol{r}), \tag{A 8}$$

which can be proven by using the previous theorem. From (A 3) and (A 5), with $\boldsymbol{u}$ as the $\boldsymbol{\sigma}$-variable, on integration over $\boldsymbol{a}$, we write

$$P_E(\boldsymbol{u} \,|\, t, \boldsymbol{r}) = \int P_L(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{x}' \,|\, t, \boldsymbol{a}, \boldsymbol{a}') \,\mathrm{d}\boldsymbol{a} \,\mathrm{d}\boldsymbol{a}'$$

from the definition (A 3), and eliminating the variable $\boldsymbol{a}$:

$$= \int \langle \delta(\boldsymbol{u}_L - \boldsymbol{u})\, \delta(\boldsymbol{y} - \boldsymbol{x} - \boldsymbol{a})\, \delta(\boldsymbol{y}' - \boldsymbol{x}' - \boldsymbol{a}) \rangle \,\mathrm{d}\boldsymbol{a}\,\mathrm{d}\boldsymbol{r}_0$$

$$= \int \langle \delta(\boldsymbol{u}_L - \boldsymbol{u})\, \delta(\boldsymbol{r}_L - \boldsymbol{r}) \rangle \,\mathrm{d}\boldsymbol{r}_0 = \int P_L(\boldsymbol{u}, \boldsymbol{r} \mid t, \boldsymbol{r}_0)\,\mathrm{d}\boldsymbol{r}_0. \qquad \square$$

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